We 1st Prove Theorem 4.5 & Theorem 4.6 on roots.
(Ziller's notes).

$\mathfrak{g} = k \otimes \mathbb{C}$ $\quad k$ is a Semi-Simple Compact Lie algebra.

(we do not need this for many arguements below. Namely holds for general Semi-simple Lie algebra)

$\eta = t \otimes \mathbb{C}$, $\quad t$ is the maximal Abelian algebra (corresponds to a maximum torus)

η is called a Cartan Sub-algebra.

[satisfies two properties (i) maximum Abelian
(ii) ad_H is Semi-simple $\forall\, H \in \eta$]

If η' is also Abelian, $\eta \subset \eta' \Rightarrow \forall\, X + iY \in \eta'$ $\quad \underbrace{[t, X+iY]}_{=0} = [t, X] + i[t, Y]$

$\Rightarrow X, Y \in t \Rightarrow \eta' = \eta$ by definition.

1

Theorem 4.5 Let Δ be the set of roots. Then

(a) $\mathfrak{g} = \eta \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ $\quad \mathfrak{g}_0 = \eta$ since if X satisfies $ad_H(X) = 0\ \forall H \in \eta \Rightarrow X \in \eta$ $\quad \mathfrak{g}_\alpha := \{ X \mid ad_H(X) = \alpha(H) X \}$ Namely eigen-space

(b) If $\alpha, \beta \in \Delta \cup \{0\}$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(c) $\{\alpha \mid \alpha \in \Delta\}$ Spans η^*

(d) If $\alpha, \beta \in \Delta \cup \{0\}$, $\alpha + \beta \neq 0$, $\quad B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$

(e) $\alpha \in \Delta \Rightarrow -\alpha \in \Delta$

(f) $B|_\eta$ is non-degenerate.

{ $\mathfrak{g}_\alpha = 0$ if, there is No linear functional $\alpha(H)$ Satisfies the condition $\stackrel{\not\exists}{0}$
Only finite many Such $\alpha(H)$, with $\alpha(H) \neq 0$

Pf. (a) By the fact that one can Simultaneously diagonalize all $ad_H : \mathfrak{g} \to \mathfrak{g}$ with $H \in \eta \Rightarrow$
$\exists\ \alpha_1 \cdots \alpha_n$ such that $ad_H(X_k) = \alpha_k(H) X_k$
Some of them are zero functional. the corresponding $\text{Span}\{X_i\}$ $= \eta$.

The reason is: (i) $\eta \subset \mathrm{Span}\{X_k \quad ad_H(X_k) = 0\}$

(ii) If X satisfies $[H, X] = 0 \quad \forall H \in \eta$

$X = Z + \sqrt{-1}W$ has $Z, W \in t$ hence $X \in \eta$.

$\mathfrak{g}_\alpha$ now are collections of eigenvectors (& span) of nonzero eigenvalues. $\mathfrak{g} = \eta \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ follows

by the definition of $\mathfrak{g}_\alpha$ & Δ.

(b) $X \in \mathfrak{g}_\alpha \quad Y \in \mathfrak{g}_\beta$

$\Rightarrow \quad [H, X] = \alpha(H) X \qquad \forall H$

$[H, Y] = \beta(H) Y$

$$[H, [X, Y]] = -[X, [Y, H]] - [Y, [H, X]]$$

$$= [X, Y]\beta(H) - \alpha(H)[Y, X] = (\alpha(H) + \beta(H))[X, Y]$$

Hence the result.

(c) If $\{\alpha(H)\}_{\alpha \in \Delta}$ does not span η^*, $\Rightarrow \exists H \in \eta$ such that $\alpha(H) = 0$, $\forall H \in \eta$

This implies $\forall Y \in \mathfrak{g} \quad Y = Z + \sum X_{\alpha_i} \quad X_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$

$\Rightarrow \quad ad_H(Y) = \sum_i ad_H(X_{\alpha_i}) = \underbrace{\alpha_i(H)}_{= 0} X_{\alpha_i} = 0$

$\Rightarrow H \in \mathfrak{z}(\mathfrak{g})$.

(d) $\forall \mathfrak{g}_\gamma$ & $Z \in \mathfrak{g}_\gamma$

Proof 1:

$ad_X \, ad_Y (Z) \in \mathfrak{g}_{\alpha+\beta+\gamma}$

Hence if $\alpha + \beta \neq 0 \Rightarrow \mathfrak{g}_{\alpha+\beta+\gamma} \neq \mathfrak{g}_\gamma$

Proof 2: $B([H, X_\alpha], X_\beta) = \alpha(H) B(X_\alpha, X_\beta)$

$= -B(X_\alpha, [H, X_\beta]) = -\beta(H) B(X_\alpha, X_\beta)$

$\Rightarrow (\alpha+\beta)(H) B(X_\alpha, X_\beta) = 0$

if $\alpha + \beta \neq 0 \Rightarrow$

$B(X_\alpha, X_\beta) = 0$

proof 3.

$ad_{X_\alpha} \cdot ad_{X_\beta}$ is nilpotent by (b).

For $Z \in \mathfrak{h}$ $\quad ad_Y(Z) = -\beta(Z)\, Y$

$\Rightarrow \quad ad_X\, ad_Y(Z) = -\beta(Z)\,[X, Y] \in \underset{\text{Could be zero if } \alpha+\beta \notin \Delta.}{\mathfrak{g}_{\alpha+\beta}} \neq \mathfrak{h}$

$\Rightarrow$ $tr(ad_X\, ad_Y) = 0$, by the definition.

(e) If $-\alpha \notin \Delta \Rightarrow \forall \beta \neq -\alpha$, $\mathfrak{g}_\beta$ is orthogonal to $\mathfrak{g}_\alpha$ w.r.t B.

$\Rightarrow$ $\mathfrak{g}_\gamma$ $\gamma \in \Delta$ all orthogonal to $\underbrace{X \in \mathfrak{g}_\alpha}$ w.r.t B.

this include $\mathfrak{h}$ since $\mathfrak{h} = \mathfrak{g}_0$

$\Rightarrow$ $\alpha + 0 = \alpha \neq 0$ $\quad \mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_0$

$\Rightarrow$ $B(X, Z) = 0 \quad \forall Z \in \mathfrak{g}$

A contradiction on B is non-degenerate.

(f) B is non-degenerate. &

$B(\mathfrak{h}, \mathfrak{g}_\alpha) = 0 \quad \forall \alpha \in \Delta$

$\Rightarrow$ $B|_{\mathfrak{h}}$ must be non-degenerate

otherwise $\exists$ $X \in \mathfrak{h}$ $\quad B(X, Z) = 0 \quad \forall Z \in \mathfrak{h}$

Then $\forall Y \in \mathfrak{g}$ $\quad Y = Z_0 + \sum X_\alpha$

$\underset{0}{\underset{\|}{B(X, Z_0)}} + \sum \underset{0}{\underset{\|}{B(X, X_\alpha)}} = 0 \Rightarrow X \in Rad(B)$

② Now $\forall\ \alpha(H) \neq 0$ — a linear functional of η.

$\exists\ H_\alpha \in \eta$ $\quad B(H_\alpha, H) = \alpha(H)$. (*)

$H = \sum h^i e_i$

$\alpha(H) = \sum h^i \alpha_i$

$\alpha_i = \alpha(e_i)$

$H_\alpha = \sum y^i e_i$, $\quad y^i h^j B_{ij} = \alpha_j h^j$

$\Leftrightarrow \quad y^i B_{ij} = \alpha_j$

since (B_{ij}) is non-degenerate.

Such a solution exists. It is also unique!

$$B(H_\alpha, H_\beta) = \alpha(H_\beta) = \beta(H_\alpha), \quad \text{if } \alpha, \beta \in \Delta.$$

Thm 4.6 ⓐ $[X_\alpha, X_{-\alpha}] = B(X_\alpha, X_{-\alpha}) \cdot H_\alpha$

$X_\alpha \in \mathfrak{g}_\alpha$
$X_{-\alpha} \in \mathfrak{g}_{-\alpha}$

ⓑ $B(H_\alpha, H_\alpha) \neq 0$

ⓒ $\dim \mathfrak{g}_\alpha = 1$.

Pf: ⓐ $B([X_\alpha, X_{-\alpha}], H) = -B([H, X_\alpha], X_\alpha)$

$\underset{-\mathrm{ad}\, X_\alpha}{\overset{\shortparallel}{}} \ X_{-\alpha}$

$= - -\alpha(H)\ B(X_{-\alpha}\ X_\alpha)$

$= +B(H_\alpha, H)\ B(X_{-\alpha}, X_\alpha)$

$= B(B(X_{-\alpha}, X_\alpha) H_\alpha, H), \forall H.$

$\Rightarrow \quad [X_\alpha, X_{-\alpha}] = B(X_{-\alpha}, X_\alpha) H_\alpha$

(Since $[X_\alpha, X_{-\alpha}] \in \eta$ by ⓑ of Thm 4.5)

& B is non-degenerate on η $\Rightarrow$ ⓐ.

If such $X_{-\alpha}$ does Not exist ⓓ of Thm 4.5 $\Rightarrow$ $X_\alpha \in \mathrm{Rad}(B)$

Choose one $X_{-\alpha}$ such that $B(X_\alpha, X_{-\alpha}) \neq 0$

By scaling we may choose $X_{-\alpha}$ $B(X_\alpha, X_{-\alpha}) = 1$

For ⓑ we first need a claim.

Digression: two linear algebra results.

Thm A: If $\mathcal{F}$ is a family of diagonalizable linear operators over V - a vector space. Then $\exists$ basis of V. every $f \in \mathcal{F}$ is diagonal.
Hoffman & Kunze: LH thm 8 of p207.

Thm B: Every normal operator is diagonalizable. Hoffman-Kunze P.317. Coro

T normal. $\Leftrightarrow TT^* = T^*T$.

$\boxed{\text{Claim: Given } \alpha \in \Delta, \exists \beta \in \Delta, B(H_\alpha, H_\beta) \neq 0.}$

Thm 4.5 (d) also works for η $\quad B(\underset{0}{\underset{\parallel}{[H, H']}}, X_\alpha) = -B(H', \alpha(H)(X_\alpha))$
$= -\alpha(H) \; B(H', X_\alpha)$

Hence if we pick H $\alpha(H) \neq 0 \Rightarrow B(H', X_\alpha) = 0$

$B(\eta, \mathfrak{g}_\alpha) = 0 \quad B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha(H) + \beta(H) \neq 0$.

Hence if $B(H_\alpha, H_\beta) = 0 \quad \forall H_\beta \; \beta \in \Delta \Rightarrow B(H_\alpha, \eta) = 0$ which is

Contradictory to $B|_\eta$ is non-degenerate.

$\exists X_{-\alpha} \in \mathfrak{g}_{-\alpha} \quad B(X_\alpha, X_{-\alpha}) \neq 0$ otherwise $B(X_\alpha, Y) = 0 \; \forall Y \in \mathfrak{g}_{-\alpha}$

$B(X_\alpha, \eta) = 0 \quad B(X_\alpha, X_\beta) = 0$ if $\alpha + \beta \neq 0$ Hence $B(X_\alpha, \mathfrak{g}_\beta) = 0$ $\forall \beta \neq -\alpha$

$\Rightarrow B(X_\alpha, \mathfrak{g}) = 0$ A Contradiction!

Now if $B(H_\alpha, H_\beta) = 0 \quad \forall H_\beta \in \Delta$

$\Rightarrow B(H_\alpha, Z) = 0 \; \forall Z \in \eta \Rightarrow H_\alpha = 0$, A Contradiction!

Now in (a), we may choose $X_{-\alpha}$ Such that $B(X_\alpha, X_{-\alpha}) = 1$

Consider the action of ad_{H_α} on $S = \sum_{n \in \mathbb{Z}} \mathfrak{g}_{\beta + n\alpha}$

Since $H_\alpha = [X_\alpha, X_{-\alpha}] \Rightarrow$

$$\operatorname{tr}(\operatorname{ad}_{H_\alpha}|_V) = \operatorname{tr}([\operatorname{ad}_{X_\alpha}\ \operatorname{ad}_{X_{-\alpha}}]) = 0$$

namely $\operatorname{ad}_{H_\alpha}|_{\mathfrak{g}_\gamma} = \gamma(H_\alpha)\,\mathrm{id}$

&

$$[H_\alpha\ \ \mathfrak{g}_\gamma] = \gamma(H_\alpha)\ \mathfrak{g}_\gamma \qquad \forall\ \mathfrak{g}_\gamma$$

[Here we have used implicitly S is invariant under $\operatorname{ad}_{X_\alpha}$ $\operatorname{ad}_{X_{-\alpha}}$ & $\operatorname{ad}_{H_\alpha}$]

$$\Rightarrow \operatorname{tr}(\operatorname{ad}_{H_\alpha}) = (\beta+n\alpha)\cdot(H_\alpha)\ \dim(\mathfrak{g}_{\beta+n\alpha})$$

$$\Rightarrow \underbrace{-\beta(H_\alpha)}_{\neq 0}\sum_n \underbrace{\dim(\mathfrak{g}_{\beta+n\alpha})}_{>0} = \alpha(H_\alpha)\sum_n n\dim(\mathfrak{g}_{\beta+n\alpha}) \Rightarrow \alpha(H_\alpha) \neq 0.$$

Namely $B(H_\alpha, H_\alpha) \neq 0$.

(c) Apply a similar arguement to

$$S = \mathbb{C}X_{-\alpha} + \mathbb{C}H_\alpha + \sum_{n\geq 1}\mathfrak{g}_{n\alpha}$$

$\operatorname{ad}_{H_\alpha}$ invariant

$\operatorname{ad}X_\beta(\mathfrak{g}_\gamma) \in \mathfrak{g}_{\beta+\gamma}$

$\operatorname{ad}_{X_\alpha}$ invariant

& $\operatorname{ad}_{X_{-\alpha}}$ invariant.

Apply the Computation of $\operatorname{tr}(\operatorname{ad}_{H_\alpha}|_S)$.

$$\Rightarrow \operatorname{ad}_{H_\alpha}(X_{-\alpha}) = (-\alpha)(H_\alpha)\,X_{-\alpha}$$

$$\mathrm{ad}_{H_\alpha}(H_\alpha) = 0$$

$$\mathrm{ad}_{H_\alpha}\Big|_{\mathfrak{g}_{n\alpha}} = n\alpha(H_\alpha)\,\mathrm{id}.$$

$$\Rightarrow \quad 0 = -B(H_\alpha, H_\alpha) + B(H_\alpha, H_\alpha)\sum_{n\geq 1} n \dim(\mathfrak{g}_{n\alpha})$$

Hence $\quad 0 = -1 + \sum_{n\geq 1} n\dim(\mathfrak{g}_{n\alpha})$

$$\Rightarrow \quad \dim(\mathfrak{g}_\alpha) = 1 \quad \& \quad \dim(\mathfrak{g}_{n\alpha}) = 0 \quad \forall\, n \geq 2.$$

---

③ Remarks:

$a: G \longrightarrow \mathrm{Aut}(G) \qquad a_g$ or $a(g): h \mapsto g h g^{-1}$

$\mathrm{Ad}: G \longrightarrow \mathrm{Aut}(\mathfrak{g}) \qquad \mathrm{Ad}_g = d a_g \qquad X \to \frac{d}{dt}\Big|_{t=0} (g \exp(tX) g^{-1})$

$\mathrm{ad}: \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}) \qquad \mathrm{ad}_X := d(\mathrm{Ad}) \qquad \mathrm{ad}_X(Y) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}_{\exp(tX)}(Y)$

$$\varphi(\exp(tX)) = \exp(t\, d\varphi(X)).$$

E.g

If $[X\ Y] = 0 \qquad \exp tX \exp(sY)\exp(-tX) = \exp(sY).$

$$\mathrm{LHS} = a_{\exp(tX)}(\exp(sY)) = \exp(s\,\mathrm{Ad}_{\exp(tX)}(Y))$$

$$\mathrm{Ad}_{\exp(tX)}(Y) = e^{t\,\mathrm{ad}(X)}(Y) = Y \quad \text{by } [X\ Y] = 0$$

Hence $\quad \mathrm{LHS} = \exp(sY).$

Namely in order to understant the group structure of G we reduce it to understant $ad_X : \mathfrak{g} \to \mathfrak{g}$ linear maps. which is the 2nd derivative of the group structure to some degree.

$$Ad: G \to GL(\mathfrak{g})$$

$$ad: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

ad is a faithful representation if $\mathfrak{z}(\mathfrak{g}) = \{0\}$

Namely if $ad_X = 0 \implies X = 0.$

$$\Updownarrow$$

$$[X, Y] = 0 \ \forall Y$$

This proves $\mathfrak{g} \simeq ad(\mathfrak{g}) \subset \mathfrak{gl}(n, \mathbb{C})$. Namely the Ado's theorem for this special case.

What is a root? $\sim$ "quantum eigenvalue"

$\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ - a complex semi-simple Lie algebra.

$\exists \{X_i\}$ $\forall H \in \mathfrak{h}$ ad_H is diagonal.

namely

$$ad_H(X_i) = \lambda_i(H) X_i$$

It can be checked, λ_i is linear. Hence $\lambda_i \in \mathfrak{h}^*$

Such an $\lambda_i(H)$ is called a root.

$\lambda_i(H)$ solves $\det(ad_H - \lambda\, id) = 0.$

root space $\mathfrak{g}_\alpha = \{X \mid ad_H(X) = \alpha(H) X\}$. Namely the eigenspaces for the eigenvalue α.